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## COMMENT

### Bound states of the potential $V(r) = -Ze^2/(r + \beta)$

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**Abstract.** Energy eigenvalues of various states from  $n = 1$  to  $n = 4$  have been calculated in the framework of the shifted  $1/N$  expansion method. Results for  $s$  states are compared with those of Mehta and Patil for small  $\beta$  values. A new approximate formula has been developed for a quick estimate of energy eigenvalues for any angular momentum state.

The problem of determining the energy eigenvalues of the cutoff Coulomb potential

$$V(r) = -\frac{Ze^2}{r + \beta} \quad \beta > 0 \quad (1)$$

has been of some interest in the past [1]. As pointed out in [1], this potential may serve as an approximation to the potential due to a smeared charge distribution rather than a point charge and may be appropriate for describing mesonic atoms.

For  $s$  states the Schrödinger equation for the potential (1) yields exact closed solutions. But the actual determination of the eigenvalues from the zeros of the confluent hypergeometric function is not easy [1]. Besides, for non-zero angular momentum states, this potential does not admit closed solutions. For such problems, therefore, one has to resort to approximate methods. Mehta and Patil [1] carried out a dispersion theoretic study of the  $s$  wave bound states of this potential for small values of the parameter  $\beta$  and concluded that the energy eigenvalue  $E(\beta)$  has a logarithmic branch cut at  $\beta = 0$  which precludes a strict perturbation series for  $E(\beta)$  around  $\beta = 0$ .

To make progress, we must make an expansion of some kind. In a number of bound-state problems [2-21] in non-relativistic quantum mechanics,  $1/N$ , where  $N$  is a very large number of spatial dimension, has proved to be a very suitable expansion parameter. It is interesting to note in this context that in quantum chromodynamics one also encounters the apparent absence of a relevant expansion parameter. The large- $N$  expansion technique has yielded useful approximate solutions for problems in QCD. Here, of course,  $N$  is the number of colours [4]. In this paper we investigate the eigenvalues of the potential (1) using a variant of the  $1/N$  expansion method. For spherically symmetric potentials the usual choice of the expansion parameter has been  $1/K$  with  $K = N + 2l$ , instead of  $1/N$ . The convergence of this method, however, is rather slow. In order to improve upon the convergence Sukhatme and Imbo [13] modified the standard method and introduced what has come to be known as the shifted  $1/N$  expansion method, in which the expansion parameter is  $1/\bar{K}$ , with  $\bar{K} = N + 2l - a$  and  $a$  is a free parameter. The choice of this shift parameter  $a$  was motivated by requiring agreement between the  $1/\bar{K}$  expansions and the exact analytic

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results for the harmonic oscillators and Coulomb potentials. In a sense, the shift  $a$  provides a physically motivated resummation of the perturbation series for the energy eigenstates, improving its convergence.

The reduced radial Schrödinger equation in  $N$  dimensions in terms of the shifted variable  $\bar{K}$  is

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\bar{K}^2 \hbar^2}{8mr^2} \left(1 - \frac{1-a}{\bar{K}}\right) \left(1 - \frac{3-a}{\bar{K}}\right) + V(r) \right] u(r) = Eu(r) \tag{2}$$

where  $V(r)$  is given by (1). The procedure for determining the energy eigenvalues from (2) is given in the paper by Imbo *et al* [14]. Hence, for the sake of brevity, we present here only the final expressions required to obtain the bound-state energies. The eigenvalues are given by

$$E_{nl} = \frac{\bar{K}^2}{r_0^2} \left[ \left( \frac{\hbar^2}{8m} + \frac{r_0^2 V(r_0)}{\bar{K}^2} \right) + \frac{\gamma^{(1)}}{\bar{K}^2} + \frac{\gamma^{(2)}}{\bar{K}^3} + O\left(\frac{1}{\bar{K}^4}\right) \right] \tag{3}$$

where

$$\bar{K} = N + 2l - a \tag{4a}$$

$$a = 2 - 2(2n_r + 1) \frac{m\omega}{\hbar} \quad n_r = 0, 1, 2, \dots \tag{4b}$$

$$\omega = \frac{\hbar}{2m} \left( 3 + \frac{r_0 V^{(2)}(r_0)}{V^{(1)}(r_0)} \right)^{1/2} \tag{4c}$$

$V^{(n)}(x)$  being the  $n$ th derivative of  $V(x)$ .  $r_0$  is determined from the equation

$$N + 2l - 2 + (2n_r + 1) \left( 3 + \frac{r_0 V^{(2)}(r_0)}{V^{(1)}(r_0)} \right)^{1/2} = \left( \frac{4mr_0^3 V^{(1)}(r_0)}{\hbar^2} \right)^{1/2} \tag{4d}$$

$\gamma^{(1)}$  and  $\gamma^{(2)}$  are given by

$$\begin{aligned} \gamma^{(1)} = & \frac{\hbar^2}{8m} (1-a)(3-a) + (1+2n_r)\tilde{\epsilon}_2 + 3(1+2n_r+2n_r^2)\tilde{\epsilon}_4 \\ & - \frac{1}{\hbar\omega} [\tilde{\epsilon}_1^2 + 6(1+2n_r)\tilde{\epsilon}_1\tilde{\epsilon}_3 + (11+30n_r+30n_r^2)\tilde{\epsilon}_3^2] \end{aligned} \tag{5}$$

$$\begin{aligned} \gamma^{(2)} = & (1+2n_r)\tilde{\delta}_2 + 3(1+2n_r+2n_r^2)\tilde{\delta}_4 + 5(3+8n_r+6n_r^2+4n_r^3)\tilde{\delta}_6 \\ & - (1/\hbar\omega)[(1+2n_r)\tilde{\epsilon}_2^2 + 12(1+2n_r+2n_r^2)\tilde{\epsilon}_2\tilde{\epsilon}_4 + 2(21+59n_r \\ & + 51n_r^2 + 34n_r^3)\tilde{\epsilon}_4^2 + 2\tilde{\epsilon}_1\tilde{\delta}_1 + 6(1+2n_r)\tilde{\epsilon}_1\tilde{\delta}_3 + 30(1+2n_r+2n_r^2)\tilde{\epsilon}_1\tilde{\delta}_5 \\ & + 6(1+2n_r)\tilde{\epsilon}_3\tilde{\delta}_1 + 2(11+30n_r+30n_r^2)\tilde{\epsilon}_3\tilde{\delta}_3 \\ & + 10(13+40n_r+42n_r^2+28n_r^3)\tilde{\epsilon}_3\tilde{\delta}_5] \\ & + [1/(\hbar\omega)^2][4\tilde{\epsilon}_1^2\tilde{\epsilon}_2 + 36(1+2n_r)\tilde{\epsilon}_1\tilde{\epsilon}_2\tilde{\epsilon}_3 + 8(11+30n_r+30n_r^2)\tilde{\epsilon}_2\tilde{\epsilon}_3^2 \\ & + 24(1+2n_r)\tilde{\epsilon}_1^2\tilde{\epsilon}_4 + 8(31+78n_r+78n_r^2)\tilde{\epsilon}_1\tilde{\epsilon}_3\tilde{\epsilon}_4 \\ & + 12(57+189n_r+225n_r^2+150n_r^3)\tilde{\epsilon}_3^2\tilde{\epsilon}_4] \\ & - [1/(\hbar\omega)^3][8\tilde{\epsilon}_1^3\tilde{\epsilon}_3 + 108(1+2n_r)\tilde{\epsilon}_1^2\tilde{\epsilon}_3^2 + 48(11+30n_r+30n_r^2)\tilde{\epsilon}_1\tilde{\epsilon}_3^3 \\ & + 30(31+109n_r+141n_r^2+94n_r^3)\tilde{\epsilon}_3^4] \end{aligned} \tag{6}$$

with

$$\tilde{\epsilon}_j = \frac{\epsilon_j}{(2m\omega/\hbar)^{j/2}} \quad \tilde{\delta}_j = \frac{\delta_j}{(2m\omega/\hbar)^{j/2}} \quad j = 1, 2, \dots$$

It is easy to see from (3) that for  $\beta \rightarrow 0$ ,  $\gamma^{(1)}$  and  $\gamma^{(2)}$  vanish identically and one recovers the Coulomb result for  $N = 3$

$$E_{nl} = -\frac{Z^2 e^2}{2n^2 a_0} \quad n = n_r + l + 1.$$

Once  $r_0$  is determined from (4d) for the potential (1) in the three-dimensional world ( $N = 3$ ), the energy eigenvalues can readily be calculated from (3). We have calculated in atomic units ( $e = m = \hbar = 1$ ) the energy levels for the states from  $n = 1$  to  $n = 4$ , which we display in tables 1 and 2. With  $\beta$  non-zero, the accidental degeneracy of

**Table 1.** Comparison of the energy eigenvalues. Column A is the result of the shifted  $1/N$  expansion, equation (3); column B is the result of Mehta and Patil, equation (8); column C is the result according to our approximation, equation (10).

$\beta$	$-E_{1s}$			$-E_{4l}$	
	A	B	C	A	C
0.0	0.500 000	0.500 000	0.500 000	0.031 250	0.031 250
0.0001	0.499 812	0.499 800	0.499 799	0.031 249	0.031 249
0.001	0.498 138	0.498 027	0.498 009	0.031 245	0.031 245
0.01	0.482 534	0.481 842	0.480 947	0.031 205	0.031 205
0.1	0.386 471	0.392 103	0.364 745	0.030 813	0.030 810
1.00	0.180 023	-1.500 000	0.125 000	0.027 588	0.027 359
2.0	0.124 858	-14.590 355	0.076 201	0.024 971	0.024 353
3.0	0.098 123		0.055 555	0.022 959	0.021 957
4.0	0.081 812		0.043 966	0.021 341	0.020 000
5.0	0.070 629		0.036 492	0.019 999	0.018 371
10.0	0.043 422		0.020 000	0.015 576	0.013 096
15.0	0.032 061		0.013 888	0.013 007	0.010 204

**Table 2.** Energy eigenvalues ( $-E_{nl}$ ) as a function of  $\beta$  for different excited states in atomic units.

$\beta$	2s	2p	3s	3p	3d	4s	4p	4d
0.0	0.125 00	0.125 00	0.055 55	0.055 55	0.055 55	0.031 25	0.031 25	0.031 25
0.0001	0.124 98	0.124 99	0.055 55	0.055 55	0.055 55	0.031 25	0.031 25	0.031 25
0.001	0.124 83	0.124 92	0.055 52	0.055 53	0.055 54	0.031 24	0.031 24	0.031 24
0.01	0.123 33	0.124 18	0.055 18	0.055 32	0.055 41	0.031 12	0.031 16	0.031 19
0.1	0.111 27	0.117 54	0.052 13	0.053 41	0.054 14	0.030 04	0.030 41	0.030 66
1.0	0.069 61	0.082 84	0.037 44	0.041 86	0.045 01	0.023 57	0.025 33	0.026 65
2.0	0.054 38	0.065 72	0.030 85	0.035 28	0.038 78	0.020 07	0.022 09	0.023 72
3.0	0.045 89	0.055 54	0.026 96	0.031 05	0.034 48	0.017 90	0.019 92	0.021 59
4.0	0.040 23	0.048 59	0.024 26	0.028 00	0.031 25	0.016 36	0.018 29	0.019 95
5.0	0.036 07	0.043 45	0.022 23	0.025 65	0.028 70	0.015 18	0.017 01	0.018 62
10.0	0.024 80	0.029 44	0.016 34	0.018 74	0.021 02	0.011 66	0.013 06	0.014 37
15.0	0.019 44	0.022 82	0.013 32	0.015 17	0.016 97	0.009 76	0.010 82	0.011 98

the Coulomb potential is immediately lifted. For small values of  $\beta$ , it seems from the tables that the degeneracy is not removed. If, however, we display the results with more digits in the decimal place, it is immediately clear that the degeneracy has indeed been lifted. As expected on physical grounds, the energy of any particular state decreases as  $\beta$  increases. We also observe that for any particular  $n$ , the state with higher angular momentum is more tightly bound than that with lower  $l$ . For the potential (1) this is consistent with the criterion of Grosse and Martin [22], who showed that for the potential  $V(r) = -1/r + U(r)$ , if

$$\frac{d}{dr} \left( r^2 \frac{dU}{dr} \right) > 0 \quad E_{nl} > E_{n'l'} \text{ for } l < l' \quad (7)$$

and if

$$\frac{d}{dr} \left( r^2 \frac{dU}{dr} \right) < 0 \quad E_{nl} < E_{n'l'} \text{ for } l < l'.$$

For small  $\beta$ , Mehta and Patil [1] obtained for  $s$  states the energy eigenvalues

$$E_{n0} = -\frac{Z^2}{2n^2} \left[ 1 - \frac{4\beta Z}{na_0} - \frac{8\beta^2 Z^2 \ln \beta}{na_0^2} \right] \frac{e^2}{a_0} + O(\beta^3). \quad (8)$$

Since (8) does not give energy eigenvalues for non-zero angular momentum states, we have developed an approximate formula for calculating energy eigenvalues for all angular momentum states, zero or non-zero. For small  $\beta$ , the potential (1) may be expanded as

$$V(r) \approx -\frac{Ze^2}{r} + \frac{Ze^2\beta}{r^2}. \quad (9)$$

The eigenvalues of (9) may be readily calculated obtaining

$$E_{nl} = -\frac{Z^2}{2(n+L-l)^2} \frac{e^2}{a_0} \quad n = 1, 2, 3 \quad (10)$$

with

$$L = -0.5 + \sqrt{(l + \frac{1}{2})^2 + (2Z\beta/a_0)}.$$

In table 1 we also display the results for  $1s$  and  $4f$  states as calculated from (8) and (10) for the purpose of comparison. For small  $\beta$ , both equations (8) and (10) agree very well with that obtained in the framework of the shifted  $1/N$  expansion, i.e. from (3), the agreement with (10) being for all  $l$  values. Even for  $\beta$  as large as 15, the result obtained by (10) for the  $1s$  state deviates from that obtained from (3) by about 54%, whereas (8) cannot even provide the binding. As  $l$  increases, the agreement between (10) and (3) gets progressively better, the deviation being about 22% for the  $4f$  state at  $\beta = 15$ . Equation (10) may, therefore, be used for a quick estimate of the energy eigenvalues of the cutoff Coulomb potential (1).

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